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Human and Robot Interaction Laboratory
Advanced Robotics Courses
Dynamic of Serial Manipulators

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Problem 1

Solve the inverse dynamic problem of the RR mechanism, shown in Fig. 1.1.

$$[I_1]_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{m_1 {l_1}^2}{12} & 0 \\ 0 & 0 & \frac{m_1 {l_1}^2}{12} \end{bmatrix}; \qquad [I_2]_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{m_2 {l_2}^2}{12} & 0 \\ 0 & 0 & \frac{m_2 {l_2}^2}{12} \end{bmatrix}$$

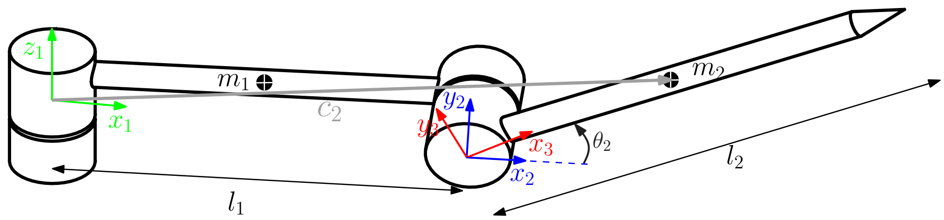

Figure 1.1: RR Serial Manipulator, Problem 1.

- Kinetic Energy

$$T = ? \Rightarrow T = \sum_{i=1}^{2} T_i = \sum_{i=1}^{2} (\frac{1}{2}\dot{\vec{c}_i}^T m_i \dot{\vec{c}_i} + \frac{1}{2}\vec{\omega}_i^T I_i \vec{\omega}_i)$$

  - Kinetic energy of the first Body (m_1, l_1)

$$T_1 = \frac{1}{2}\dot{\vec{c}_1}^T m_1 \dot{\vec{c}_1} + \frac{1}{2}\vec{\omega}_1^T I_1 \vec{\omega}_1$$

$$\vec{\omega}_{R_1} = \dot{\theta}_1 \vec{k}_1 = \dot{\theta}_1 \vec{j}_2$$

$$\vec{c_1} = \frac{l_1}{2}\vec{i_2} \longrightarrow \dot{\vec{c}_1} =]\dot{\vec{c}_1}[+ \omega_{R_1} \times c_1 = \dot{\theta}_1 \vec{j}_2 \times \frac{l_1}{2}\vec{i}_2 = -\frac{l_1}{2}\dot{\theta}_1 \vec{k}_2$$

(I)

$$\dot{\vec{c}_1} = -\frac{l_1}{2}\dot{\theta}_1 \vec{k}_2$$

(II)

$$[\omega_1]_2 = \dot{\theta}_1 \vec{j}_2$$

$$(I)\&(II) \implies T_1 = \frac{1}{2} m_1 (\frac{{l_1}^2}{4}\dot{\theta}_1^2) + \frac{1}{2}\begin{bmatrix} 0 \\ \dot{\theta}_1 \\ 0 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{m_1 {l_1}^2}{12} & 0 \\ 0 & 0 & \frac{m_1 {l_1}^2}{12} \end{bmatrix} \begin{bmatrix} 0 \\ \dot{\theta}_1 \\ 0 \end{bmatrix}$$
$$\implies T_1 = \frac{1}{8} m_1 {l_1}^2 {\dot{\theta}_1}^2 + \frac{1}{24} m_1 {l_1}^2 {\dot{\theta}_1}^2 = \frac{1}{6} m_1 {l_1}^2 {\dot{\theta}_1}^2$$

– Kinetic energy of the second body (m_2, l_2)

$$T_2 = \frac{1}{2}\dot{\vec{c}_2}^T m_2 \dot{\vec{c}_2} + \frac{1}{2}\vec{\omega}_2^T I_2 \vec{\omega}_2$$

(I)

$$\vec{\omega}_{R_2} = \dot{\theta}_1 \vec{j}_1$$

$$\vec{c}_2 = (l_1 + \frac{l_2}{2}\cos\theta_2)\vec{i}_2 + (\frac{l_2}{2}\sin\theta_2)\vec{j}_2 \longrightarrow \dot{\vec{c}_2} = \big]\dot{\vec{c}_2}\big[+ \vec{\omega}_{R_2} \times \vec{c}_2$$

(II)

$$\omega_2 = \dot{\theta}_1 \vec{k}_1 + \dot{\theta}_2 \vec{k}_2 = \dot{\theta}_1 \vec{j}_2 + \dot{\theta}_2 \vec{k}_2 = \dot{\theta}_1(\sin\theta_2 \vec{i}_3 + \cos\theta_2 \vec{j}_3) + \dot{\theta}_2 \vec{k}_3$$

$$[\omega_2]_3 = \dot{\theta}_1 \sin\theta_2 \vec{i}_3 + \dot{\theta}_1 \cos\theta_2 \vec{j}_3 + \dot{\theta}_2 \vec{k}_3$$

$$\begin{aligned}
(I)\&(II) \implies T_2 &= \frac{1}{2}m_2\left(\frac{{l_2}^2}{4}\dot{\theta}_2^2 + (l_1 + \frac{l_2}{2}\cos\theta_2)^2 {\dot{\theta}_1}^2\right) \\
&+ \frac{1}{2}\begin{bmatrix} \dot{\theta}_1 \sin\theta_2 \\ \dot{\theta}_1 \cos\theta_2 \\ \dot{\theta}_2 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{m_2 {l_2}^2}{12} & 0 \\ 0 & 0 & \frac{m_2 {l_2}^2}{12} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \sin\theta_2 \\ \dot{\theta}_1 \cos\theta_2 \\ \dot{\theta}_2 \end{bmatrix} \\
&= \frac{1}{8}m_2 {l_2}^2 {\dot{\theta}_2}^2 + m_2\left(l_1 + \frac{l_2}{2}\cos\theta_2\right)^2 \frac{\dot{\theta}_1}{2} + \frac{1}{2}\left(\frac{m_2 {l_2}^2}{12}\dot{\theta}_1^2 \cos^2\theta_2 + \frac{m_2 {l_2}^2}{12}{\dot{\theta}_2}^2\right) \\
&= \frac{1}{6}m_2 {l_2}^2 \dot{\theta}_2^2 + \frac{1}{6}m_2 {l_2}^2 \dot{\theta}_1^2 \cos^2\theta_2 + \frac{1}{2}m_2 {l_1}^2 {\dot{\theta}_1}^2 + \frac{1}{2}m_2 l_1 l_2 {\dot{\theta}_1}^2 \cos^2\theta_2
\end{aligned}$$

– Kinetic Energy of the Mechanism

$$T = \sum_{i=1}^{2} T_i = \frac{1}{6}m_1 l_1 {\dot{\theta}_1}^2 + \frac{1}{6}m_2 l_2 {\dot{\theta}_2}^2 + \frac{1}{6}m_2 l_2 {\dot{\theta}_1}^2 \cos^2\theta_2 + \frac{1}{2}m_2 l_1^2 \dot{\theta}_1 + \frac{1}{2}m_2 l_1 l_2 {\dot{\theta}_1}^2 \cos\theta_2$$

- Potential Energy

$$V = \sum_{i=1}^{2} m_i g h_i = m_2 g \frac{l_2}{2}\sin\theta_2$$

- Euler-Lagrange Approach

I.

$$M\ddot{\vec{\theta}} + \dot{M}\dot{\vec{\theta}} - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} = \tau = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

II.

$$M = \frac{\partial^2 T}{\partial \dot{\vec{\theta}}^2} = \begin{bmatrix} \frac{\partial^2 T}{\partial \dot{\theta}_1^2} & \frac{\partial^2 T}{\partial \dot{\theta}_1 \partial \dot{\theta}_2} \\ \frac{\partial^2 T}{\partial \dot{\theta}_1 \partial \dot{\theta}_2} & \frac{\partial^2 T}{\partial \dot{\theta}_2^2} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{3}m_1{l_1}^2 + \frac{1}{3}m_2 l_2 \cos^2\theta_2 + m_2{l_1}^2 + m_2 l_1 l_2 \cos\theta_2 & 0 \\ 0 & \frac{1}{3}m_2{l_2}^2 \end{bmatrix}$$

III.

$$\dot{M} = \frac{dM}{dt} = \begin{bmatrix} -\frac{2}{3}m_2 l_2 \dot{\theta}_2 \cos\theta_2 \sin\theta_2 - m_2 l_1 l_2 \dot{\theta}_2 \sin\theta_2 & 0 \\ 0 & 0 \end{bmatrix}$$

IV.

$$\frac{\partial T}{\partial \theta} = \begin{bmatrix} \frac{\partial T}{\partial \theta_1} \\ \frac{\partial T}{\partial \theta_1} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{3}m_2{l_2}^2\dot{\theta}_1^2 \cos\theta_2 \sin\theta_2 - \frac{1}{2}m_2 l_1 l_2 \dot{\theta}_1^2 \sin\theta_2 \end{bmatrix}$$

V.

$$\frac{\partial V}{\partial \theta} = \begin{bmatrix} \frac{\partial V}{\partial \theta_1} \\ \frac{\partial V}{\partial \theta_1} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2}m_2 g l_2 \cos\theta_2 \end{bmatrix}$$

Replacing (II)-(V) in (I):

$$\tau_1 = \left(\frac{1}{3}m_1{l_1}^2 + \frac{1}{3}m_2 l_2 \cos^2\theta_2 + m_2{l_1}^2 + m_2 l_1 l_2 \cos\theta_2\right)\ddot{\theta}_1$$
$$- \left(\frac{2}{3}m_2 l_2 \dot{\theta}_2 \cos\theta_2 \sin\theta_2 - m_2 l_1 l_2 \dot{\theta}_2 \sin\theta_2\right)\dot{\theta}_1$$
$$\tau_2 = \frac{1}{3}m_2 l_2^2 \ddot{\theta}_2 - \frac{1}{3}m_2 l_2^2 \dot{\theta}_1^2 \cos\theta_2 \sin\theta_2 - \frac{1}{2}m_2 l_1 l_2 \dot{\theta}_1^2 \sin\theta_1 + m_2 g \frac{l_2}{2} \cos\theta_2$$

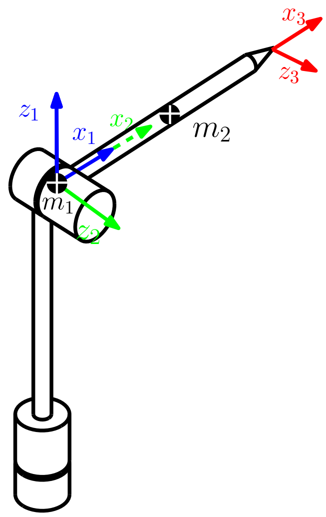

Figure 1.2: 2-DOF Serial Manipulator, Problem 2.

Problem 2

Solve the inverse dynamic problem of the RR mechanism, shown in Fig. 1.1, using the DH method.

i	a_i	b_i	α_i	θ_i
1	0	0	$\frac{\pi}{2}$	θ_1
2	l	0	0	θ_2

$$[S_1]_2 = 0; \quad [S_2]_3 = \begin{bmatrix} -\frac{l}{2} \\ 0 \\ 0 \end{bmatrix}; \quad [I_1]_2 = \begin{bmatrix} j & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & j \end{bmatrix}; \quad [I_2]_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{a}_1 = 0; \quad [a_2]_2 = \begin{bmatrix} l\cos\theta_2 \\ l\sin\theta_2 \\ 0 \end{bmatrix}$$

$$Q_1 = \begin{bmatrix} \cos\theta_1 & 0 & \sin\theta_1 \\ \sin\theta_1 & 0 & -\cos\theta_1 \\ 0 & 1 & 0 \end{bmatrix}; \quad Q_2 = \begin{bmatrix} -\cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Kinetic Energy of the 1^{st} Body (m_1)

$$T_1 = \frac{1}{2} m_1 \dot{\vec{c}}_1^T \dot{\vec{c}}_1 + \frac{1}{2} \vec{\omega}_1^T I_1 \vec{\omega}_1$$

$$\vec{c}_1 = \vec{a}_1 + \vec{s}_1 = 0$$

$$\vec{\omega}_1 = \dot{\theta}_1 \vec{k}_1 \longrightarrow [\omega_1]_2 = \dot{\theta}_1 [e_1]_2 = \dot{\theta}_1 Q_1^T [e_1]_1 = \dot{\theta}_1 \vec{j}_2$$

Note⇑

$$[e_i]_{i+1} = Q_i^T [e_i]_i = \begin{bmatrix} 0 \\ \sin\alpha_i \\ \cos\alpha_i \end{bmatrix}, [a_i]_{i+1} = Q_i^T [a_i]_i = \begin{bmatrix} a_i \\ b_i \sin\alpha_i \\ b_i \cos\alpha_i \end{bmatrix}$$

$$\Longrightarrow T_1 = \frac{1}{2} \begin{bmatrix} 0 \\ \dot{\theta}_1 \\ 0 \end{bmatrix}^T \begin{bmatrix} j & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & j \end{bmatrix} \begin{bmatrix} 0 \\ \dot{\theta}_1 \\ 0 \end{bmatrix} = \frac{1}{2} \dot{\theta} j \tag{1.1}$$

- Kinetic Energy of the 2^{nd} Body (m_2)

$$T_2 = \frac{1}{2} m_2 \dot{\vec{c}}_2^{\,T} \dot{\vec{c}}_2 + \frac{1}{2} \vec{\omega}_2^T I_2 \vec{\omega}_2$$

$$[\vec{c}_2]_2 = [\vec{a}_1]_1 + [\vec{a}_2]_2 + [\vec{S}_2]_2$$

$$[\vec{c}_2]_2 = [\vec{a}_2]_2 + Q_2 [S_2]_3 = \frac{l}{2} \begin{bmatrix} \cos\theta_2 \\ \sin\theta_2 \\ 0 \end{bmatrix}$$

$$[\omega_{R2}]_2 = [\omega_1]_2 = [\dot{\theta}_1 e_1]_2 = \dot{\theta}_1 [e_1]_2 = \dot{\theta}_1 {Q_1}^T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \dot{\theta}_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\dot{\vec{c}}_2 =]\dot{\vec{c}}_2[+ \vec{\omega}_{R2} \times \vec{c}_2 = \frac{l}{2} \begin{bmatrix} -\dot{\theta}_2 \sin\theta_2 \\ \dot{\theta}_2 \cos\theta_2 \\ 0 \end{bmatrix} + \dot{\theta}_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \times \frac{l}{2} \begin{bmatrix} \cos\theta_2 \\ \sin\theta_2 \\ 0 \end{bmatrix}$$

$$\dot{\vec{c}}_2 = \frac{l\dot{\theta}_2}{2}\begin{bmatrix} -\sin\theta_2 \\ \cos\theta_2 \\ 0 \end{bmatrix} - \frac{l\dot{\theta}_1}{2}\begin{bmatrix} 0 \\ 0 \\ \cos\theta_2 \end{bmatrix} = \frac{l}{2}\begin{bmatrix} -\dot{\theta}_2\sin\theta_2 \\ \dot{\theta}_2\cos\theta_2 \\ -\dot{\theta}_1\cos\theta_2 \end{bmatrix}$$

$$T_2 = T_2^r + T_2^t$$

$$T_2^T = \frac{1}{2}m_2\dot{\vec{c}}_2^T\dot{\vec{c}}_2 = \frac{1}{2}m_2\frac{l^2}{4}\left(\dot{\theta}_2^2 + \dot{\theta}_1^2\cos^2\theta_2\right)$$

$$T_2^r = \frac{1}{2}\vec{\omega}_2^T I_2\vec{\omega}_2$$

$$\begin{aligned}
\vec{\omega}_2 = \dot{\theta}_1\vec{e} + \dot{\theta}_2\vec{e}_2 \Rightarrow [\omega_2]_3 &= \dot{\theta}_1 Q_2^T Q_1^T [e_1]_1 + \dot{\theta}_2 Q_2^T [e_2]_2 \\
&= \dot{\theta}_1 Q_2^T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \dot{\theta}_2 Q_2^T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
&= \dot{\theta}_1 \begin{bmatrix} \cos\theta_2 & \sin\theta_2 & 0 \\ -\sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \dot{\theta}_2\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \dot{\theta}_1\sin\theta_2 \\ \dot{\theta}_1\cos\theta_2 \\ \dot{\theta}_2 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\Longrightarrow T_2^r &= \frac{1}{2}\begin{bmatrix} \dot{\theta}_1\sin\theta_2 & \dot{\theta}_1\cos\theta_2 & \dot{\theta}_2 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}\begin{bmatrix} \dot{\theta}_1\sin\theta_2 \\ \dot{\theta}_1\cos\theta_2 \\ \dot{\theta}_2 \end{bmatrix} \\
&= \frac{k}{2}\left(\dot{\theta}_2\cos^2\theta_2 + \dot{\theta}_2{}^2\right) \\
\Longrightarrow T_2^r &= \frac{k}{2}\left(\dot{\theta}_1^2\cos^2\theta_2 + \dot{\theta}_2^2\right) \\
T_2 = T_2^r + T_2^t &= \left(\frac{1}{8}m_2 l^2 + \frac{k}{2}\right)\left(\dot{\theta}_1\cos^2\theta_2 + \dot{\theta}_2^2\right) \\
\text{if} : R &= \left(\frac{1}{8}m_2 l^2 + \frac{k}{2}\right)
\end{aligned}$$

$$\Rightarrow T_2 = \frac{R}{2}\left(\dot{\theta}_1 \cos^2\theta_2 + \dot{\theta}_2^2\right) \tag{1.2}$$

$$M\ddot{\theta} + \dot{M}\dot{\vec{\theta}} - \frac{\partial T}{\partial \theta} + \frac{\partial U}{\partial \theta} = \vec{\tau} \tag{1.3}$$

$$M = \begin{bmatrix} \frac{\partial^2 T}{\partial \dot{\theta}_1^2} & \frac{\partial^2 T}{\partial \dot{\theta}_1 \partial \dot{\theta}_2} \\ \frac{\partial^2 T}{\partial \dot{\theta}_1 \partial \dot{\theta}_2} & \frac{\partial^2 T}{\partial \dot{\theta}_2^2} \end{bmatrix}$$

$$\frac{\partial T}{\partial \dot{\theta}_1} = 2R\dot{\theta}_1 \cos^2\theta_2 + J\dot{\theta}_1 \Rightarrow \frac{\partial^2 T}{\partial \dot{\theta}_1^2} = 2R\cos^2\theta_2 + J$$

$$\frac{\partial^2 T}{\partial \dot{\theta}_1 \partial \dot{\theta}_2} = \frac{\partial^2 T}{\partial \dot{\theta}_2 \partial \dot{\theta}_1} = 0, \qquad \frac{\partial T}{\partial \dot{\theta}_2} = 2R\dot{\theta}_2 \Rightarrow \frac{\partial^2 T}{\partial \dot{\theta}_2^2} = 2R$$

$$M = \begin{bmatrix} 2R\cos^2\theta_2 + J & 0 \\ 0 & 2R \end{bmatrix} \tag{1.4}$$

$$\Longrightarrow \dot{M} = \begin{bmatrix} -4R\dot{\theta}_2 \cos\theta_2 \sin\theta_2 & 0 \\ 0 & 0 \end{bmatrix} \tag{1.5}$$

$$\frac{\partial T}{\partial \theta} = \begin{bmatrix} \frac{\partial T}{\partial \theta_1} \\ \frac{\partial T}{\partial \theta_2} \end{bmatrix} = \begin{bmatrix} 0 \\ -2R\dot{\theta}_1^2 \cos\theta_2 \sin\theta_2 \end{bmatrix} \tag{1.6}$$

$$V = V_1 + V_2 = V_2 \Longrightarrow V_2 = m_2 g h_2 = m_2 g \frac{l}{2} \sin\theta_2$$

$$\frac{\partial V}{\partial \theta} = \begin{bmatrix} 0 \\ m_2 g \frac{l}{2} \cos\theta_2 \end{bmatrix} \tag{1.7}$$

Using Eqs. 1.3-1.7:

$$\tau = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} \left(J + 2R\cos^2\theta_2\right)\ddot{\theta}_1 - 4R\dot{\theta}_1\dot{\theta}_2 \sin\theta_2 \cos\theta_2 \\ 2R\ddot{\theta}_2 + 2R\dot{\theta}_1^2 \sin\theta_2 \cos\theta_2 + m_2 g \frac{l}{2} \cos\theta_2 \end{bmatrix} \tag{1.8}$$

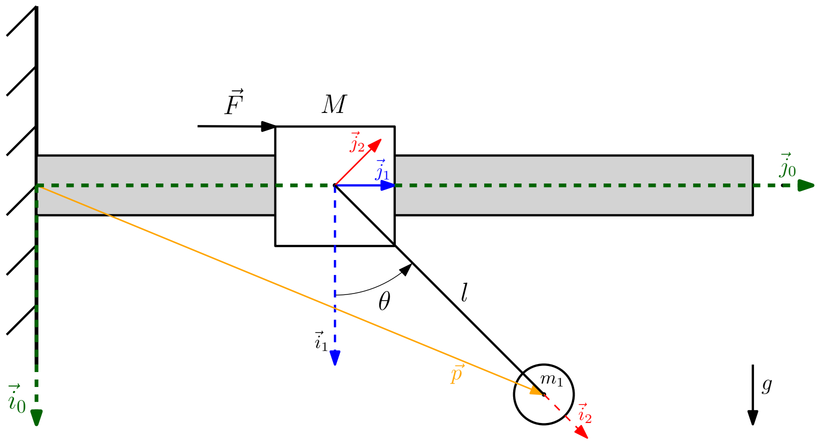

Figure 1.3: Schematic for Problem 3.

Problem 3

Find the inverse dynamic equations of the mechanism shown in Fig. 1.3

- Kinetic energy
  - Kinetic energy of M

$$T_M = \frac{1}{2} M \dot{x}^2$$

  - Kinetic energy of m

$$\vec{p} = (x + l \sin\theta)\vec{j}_0 + (l \cos\theta)\vec{i}_0$$

$$\dot{\vec{p}} = \left]\dot{\vec{p}}\right[+ \vec{\omega}_{R1} \times \vec{p} \rightarrow \dot{\vec{p}} = \left]\dot{\vec{p}}\right[$$

$$\dot{\vec{p}} = (\dot{x} + l.\dot{\theta}.\cos\theta)\vec{j}_0 + (l.\dot{\theta}.\sin\theta)\vec{i}_0$$

  - Kinetic energy of the mechanism

$$T_m = \frac{1}{2} m \left[(\dot{x} + l.\dot{\theta}\cos\theta)^2 + l^2\dot{\theta}^2 \sin^2\theta\right]$$

$$T_m = \frac{1}{2} m \left[(\dot{x}^2 + 2\dot{x}.l.\dot{\theta}.\cos\theta)^2 + l^2\dot{\theta}^2\right]$$

$$T = T_M + T_m = \frac{1}{2} M\dot{x}^2 + \frac{1}{2} m \left[\dot{c}^2 + 2\dot{x}.l.\dot{\theta}\cos\theta + l^2\dot{\theta}^2\right]$$

- Potential energy

$$V = -mgl\cos\theta$$

- Euler-Lagrange Approach

$$L = T - V \rightarrow L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left[\dot{x}^2 + 2\dot{x}l\dot{\theta}\cos\theta + l^2\dot{\theta}\right] + mgl\cos\theta$$

  - In x direction:

$$\frac{d}{dT}(\frac{\partial L}{\partial \dot{x}}) - \frac{\partial L}{\partial x} = F \rightarrow \frac{\partial L}{\partial \dot{x}} = M\dot{x} + m\dot{x} + m.l.\dot{\theta}\cos\theta$$

$$\Longrightarrow \frac{d}{dT}(\frac{\partial L}{\partial \dot{x}}) = M\ddot{x} + m\ddot{x} + m.l.\ddot{\theta}\cos\theta - m.l.\dot{\theta}^2\sin\theta,\ \frac{\partial L}{\partial x} = 0$$

$$F = (M+m)\ddot{x} + ml\ddot{x}\cos\theta - ml\dot{\theta}^2\sin\theta$$

  - In θ direction:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{L}{\theta} = \tau_\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = ml\dot{x}\cos\theta + ml^2\dot{\theta} \Longrightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml\ddot{x}\cos\theta - ml\dot{\theta}\dot{x}\sin\theta + ml^2\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -m\dot{x}l\dot{\theta}\sin\theta\sin\theta - mgl\sin\theta$$

$$\tau_\theta = ml\ddot{x}\cos\theta - ml\dot{\theta}\dot{x}\sin\theta + ml^2\ddot{\theta} + ml\dot{x}\dot{\theta}\sin\theta + mgl\sin\theta$$

$$\tau_\theta = ml\ddot{x}\cos\theta + ml^2\ddot{\theta} + mgl\sin\theta$$